

Mass-Energy and Momentum Extraction by Gravitational Wave Emission in the Merger of Two Colliding Black Holes: The Non-Head-On Case

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We examine numerically the post-merger regime of two Schwarzschild black holes in non head-on collision. Our treatment is made in the realm of non-axisymmetric Robinson-Trautman spacetimes which are appropriate for the description of the system. Characteristic initial data for the system are constructed and the Robinson-Trautman equation is integrated for these data using a numerical code based on the Galerkin spectral method which is accurate and sufficiently stable to reach the final configuration of the remnant black hole, when the gravitational wave emission ceases. The initial data contains three independent parameters, the ratio mass α of the individual colliding black holes, the boost parameter (that characterizes the initial pre-merger infalling velocity of the two black holes) and the incidence angle of collision $0 \leq \rho_0 \leq 90^\circ$. The remnant black hole is characterized by its final boost parameter, the final rest mass and scattering angle. The motion of the remnant black hole is restricted to the plane determined by the directions of the two initial colliding black holes, characterizing a planar collision. The energy-momentum fluxes carried out by gravitational waves are confined to this plane. We evaluate the efficiency of mass-energy extraction, the total energy and momentum carried out by gravitational waves and the momentum distribution of the remnant black hole for a large domain of initial data parameters. Our analysis is based on the Bondi-Sachs four momentum conservation laws. The process of mass-energy extraction is shown to be less efficient as the initial data departs from the head-on configuration. Head-on collisions ($\rho_0 = 0$) and orthogonal collisions ($\rho_0 = 90^\circ$) constitute, respectively, upper and lower bounds to the power emission and to the efficiency of mass-energy extraction. On the contrary, head-on collisions and orthogonal collisions constitute, respectively, lower and upper bounds for the momentum of the remnant. The momentum extraction and the pattern of the momentum fluxes, as a function of the incidence angle, are examined. The momentum extraction characterizes a regime of strong deceleration of the system. The angular pattern of gravitational wave signals are also examined for small mass ratios α and early times u . We show that, in the plane of collision (x, z) , the pattern is typically bremsstrahlung, corresponding to a strong deceleration regime at early times, with two dominant lobes in the forward direction of motion. Gravitational waves are also emitted outside the plane of collision, the angular distribution of which is symmetric with respect to the y -axis and therefore with a zero net momentum, consistent with the planar nature of the collision. Finally the relation between the incidence angle and the scattering angle closely approximates a relation for the inelastic collision of classical particles in Newtonian dynamics.

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I. INTRODUCTION

It is by now theoretically well established that, in the nonlinear regime of General Relativity, gravitational waves extract mass, momentum and angular momentum of the source and that the radiative transfer involved in these processes may turn out to be fundamental for the astrophysics of the collapse of stars, the formation of black holes and the collision and merger of two or more black holes. In the present paper we will examine the radiative transfer processes by gravitational wave emission in the post-merger phase of two colliding black holes, with focus on the case of non-head-on collisions, described in the realm of Robinson-Trautman (RT) spacetimes[1]. This work extends two

previous recent papers[2, 3] where we examined the case of head-on collisions. The collision and merger of two black holes are considered to be important astrophysical sources of strong gravitational emission (cf.[4] and references therein) and the relevance of the processes of generation and emission of gravitational waves in these configurations lies in the fact that the associated wave patterns will be crucial for the present efforts towards a direct detection of gravitational waves. In spite of the enormous progress achieved until now using approximation methods and numerical techniques, the information on wave form patterns and radiative transfer processes in the dynamics of gravitational wave emission is far from being complete[5].

The object of this paper is the numerical treatment of the gravitational wave production and the associated processes of radiative transfer in the non-head-on collision of two boosted Schwarzschild black holes, modeled in the context of Robinson-Trautman (RT) spacetimes. Characteristic initial data for the RT dynamics are con-

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structed that represent instantaneously two black holes in non-head-on collision. RT spacetimes present a global apparent horizon[6] so that the dynamics corresponds to a regime where the merger has already set in. During the merger part of the rest mass and kinetic energy of the two initial individual black holes are radiated away by the gravitational waves, and part will be absorbed to constitute the mass-energy of the remnant. The linear momentum of the initial system is also carried out by the gravitational waves emitted. We also examine the two regimes of momentum extraction in the post-merger phase by contemplating the time behavior of the momentum flux of the gravitational waves emitted. We use the momentum-energy conservation law in the Bondi-Sachs formulation of gravitational wave emission by bounded sources[7, 8]. Also the analysis of the energy flux carried out by gravitational waves will allow us to characterize two distinct regimes of bremsstrahlung emission for distinct domains of the initial data parameters. Several characteristics of a non-head-on collision are drastically distinct from a head-on collision, mainly for large values of the initial mass ratio parameter. The general outcome will be a remnant boosted black hole with a rest mass larger than the sum of the rest masses of the two individual initial black holes.

As compared to previous estimates of the literature, our approach differs basically in that we have adopted the characteristic surface initial data formalism, which has several advantages for the description of gravitational radiation and the construction of algorithms[9]. In this direction, an accurate code based on the Galerkin method was constructed to integrate the RT non-axisymmetric field equations. The code is accurate and highly stable for long time runs in the nonlinear regime so that we are able to reach numerically the final configuration of the system, when the gravitational emission ceases. This will allow us to describe processes of mass-energy, and momentum extraction due to gravitational radiation emission, by using physically meaningful quantities connected to initial and final configurations of the source.

We organize the paper as follows. In Sec. II we review some basic aspects of the general non-axisymmetric Robinson-Trautman spacetimes necessary for our analysis of gravitational radiation emission in the case of a non-head-on collision. In Section III we construct characteristic initial data for the non-axisymmetric RT dynamics corresponding to two boosted Schwarzschild black holes in a non-head-on collision. In Section IV we present a basic description of the numerical codes used to evolve the characteristic initial data via the RT dynamics. The codes are based on Galerkin spectral methods and the dynamical evolution of the initial data is performed and discussed, including its accuracy and stability. In Section V we discuss the planar nature of the collision and show how this property allows us to save a lot of computational effort. In Section VI the Bondi-Sachs four momentum for the non-axisymmetric RT spacetimes are introduced together with its conservation laws that will

be fundamental in the treatment of the energy and linear momentum extracted from the source by the gravitational waves emitted. Sections VII to X contain the main results of the paper, as the numerical evaluation of the efficiency of mass-energy transfer by gravitational wave emission, the conservation of energy in the processes via the Bondi mass formula and linear momentum extraction by gravitational waves. Distinct patterns of gravitational wave emission are discussed related to the range of the physical parameters of the initial data. In Section XI we summarize our results and discuss their relevance and limitations as compared to previous results in the literature. Throughout the paper we use units such that $8\pi G = c = 1$.

II. DYNAMICS OF ROBINSON-TRAUTMAN SPACETIMES. WAVE ZONE CURVATURE

RT spacetimes[1] are asymptotically flat solutions of Einstein's vacuum equations that describe the exterior gravitational field of a bounded system radiating gravitational waves. The RT metric is expressed as

$$ds^2 = \alpha^2(u, r, \theta, \phi) du^2 + 2du dr - r^2 K^2(u, \theta, \phi) \times (d\theta^2 + \sin^2 \theta d\phi^2). \quad (1)$$

where r is an affine parameter defined along the shear-free null geodesics determined by the vector field $\partial/\partial r$. Einstein equations imply that, in a suitable coordinate system,

$$\alpha^2(u, r, \theta, \phi) = \lambda(u, \theta, \phi) - \frac{2m_0}{r} + 2r \frac{K_u}{K}, \quad (2)$$

where m_0 is a positive constant, and $\lambda(u, \theta, \phi)$ is the Gaussian curvature of the surfaces ($u=\text{const}$, $r=\text{const}$) defined by

$$\lambda = \frac{1}{K^2} - \frac{(K_\theta \sin \theta / K)_\theta}{K^2 \sin \theta} + \frac{1}{\sin^2 \theta} \left(\frac{K_\phi^2}{K^4} - \frac{K_{\phi\phi}}{K^3} \right). \quad (3)$$

The remaining Einstein equations yield

$$-6m_0 \frac{K_u}{K} + \frac{1}{2K^2} \left(\frac{(\lambda_\theta \sin \theta)_\theta}{\sin \theta} + \frac{\lambda_{\phi\phi}}{\sin^2 \theta} \right) = 0. \quad (4)$$

In the above, subscripts u , θ and ϕ denote derivatives with respect to u , θ and ϕ , respectively. Eq. (4), denoted RT equation, governs the dynamics of the gravitational field which is totally contained in the metric function $K(u, \theta, \phi)$. Chrusciel and Singleton[10] established that RT spacetimes exist globally for all positive u and converge asymptotically to the Schwarzschild metric as $u \rightarrow \infty$ – this global time extension being realized for arbitrary smooth initial data.

An important feature of RT spacetimes, that establishes its radiative character, arises from the expression of its curvature tensor that in the semi-null tetrad basis

$$\begin{aligned} \Theta^0 &= du, & \Theta^1 &= (\alpha^2/2) du + dr \\ \Theta^2 &= rK d\theta, & \Theta^3 &= rK \sin \theta d\phi \end{aligned} \quad (5)$$

assumes the form

$$R_{ABCD} = \frac{N_{ABCD}}{r} + \frac{III_{ABCD}}{r^2} + \frac{II_{ABCD}}{r^3}, \quad (6)$$

where the scalar quantities N_{ABCD} , III_{ABCD} and II_{ABCD} are of the algebraic type N , III and II , respectively, in the Petrov classification of the curvature tensor[11], and r is the parameter distance along the principal null direction $\partial/\partial r$. Eq. (6) displays the peeling property[12] of the curvature tensor, showing that indeed RT is the exterior gravitational field of a bounded source emitting gravitational waves. For large r we have

$$R_{ABCD} \sim \frac{N_{ABCD}}{r}, \quad (7)$$

so that at large r the gravitational field looks like a gravitational wave with propagation vector $\partial/\partial r$. The nonvanishing of the scalars N_{ABCD} is an invariant criterion for the presence of gravitational waves, and the asymptotic region where $\mathcal{O}(1/r)$ -terms are dominant defined as the wave zone. The curvature tensor components in the above basis that contribute to N_{ABCD} , namely, to the gravitational degrees of freedom transversal to the direction of propagation of the wave, are $R_{0303} = -R_{0202} = -D(u, \theta, \phi)/r + \mathcal{O}(1/r^2)$ and $R_{0203} = -B(u, \theta, \phi)/r + \mathcal{O}(1/r^2)$ where

$$D(u, \theta, \phi) = \frac{1}{2K^2} \partial_u \left(\frac{K_{\theta\theta}}{K} - \frac{K_\theta}{K} \cot \theta - \frac{2K_\theta^2}{K^2} \right) + \frac{1}{2K^2 \sin^2 \theta} \partial_u \left(-\frac{K_{\phi\phi}}{K} + \frac{2K_\phi^2}{K^2} \right), \quad (8)$$

and

$$B(u, \theta, \phi) = \frac{1}{K^2 \sin \theta} \partial_u \left(\frac{K_{\theta\phi}}{K} - 2\frac{K_\theta K_\phi}{K^2} - \cot \theta \frac{K_\phi}{K} \right), \quad (9)$$

From (7) we can see that the functions D and B contain all the information of the angular, and time dependence of the gravitational wave amplitudes in the wave zone once $K(u, \theta, \phi)$ is given. D and B actually correspond to the two independent polarization modes of the gravitational wave, transverse to its direction of propagation at the wave zone.

The field equations present two stationary solutions, which will play a crucial role in our future discussions. The first is the Schwarzschild solution corresponding to

$$K = K_0 = \text{const}, \quad \lambda = 1/K_0^2 \quad (10)$$

and mass $M_{Schw} = m_0 K_0^3$. The second is

$$K(\theta, \phi) = \frac{K_0}{\cosh \gamma + (\mathbf{n} \cdot \hat{\mathbf{x}}) \sinh \gamma}, \quad (11)$$

where $\hat{\mathbf{x}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is the unit vector along an arbitrary direction \mathbf{x} and $\mathbf{n} = (n_1, n_2, n_3)$ is a constant unit vector (satisfying $n_1^2 + n_2^2 + n_3^2 = 1$); also K_0 and γ are constants. We note that (11) yields $\lambda = 1/K_0^2$, resulting in its stationary character. This solution can be interpreted[7] as a boosted black hole along the axis determined by the unit vector \mathbf{n} with boost parameter γ , or equivalently, with velocity parameter $v = \tanh \gamma$.

The $K(\theta, \phi)$ function (11), which depends on three parameters, is a K-transformation of the generalized Bondi-Metzner group[7] discussed by Sachs[13] (the BMS group) and represents the general form of Lorentz boosts contained in the homogeneous orthochronous Lorentz transformations of the BMS.

The Bondi mass function of this solution is given by $m(\theta, \phi) = m_0 K^3(\theta, \phi)$. The total mass-energy of this gravitational configuration is given by the Bondi mass

$$M = (1/4\pi) \int_0^{2\pi} d\phi \int_0^\pi d\theta m(\theta, \phi) \sin \theta = m_0 K_0^3 \cosh \gamma = m_0 K_0^3 / \sqrt{1 - v^2}. \quad (12)$$

The interpretation of (11) as a boosted black hole is relative to the asymptotic Lorentz frame which is the rest frame of the black hole when $\gamma = 0$.

III. CHARACTERISTIC INITIAL DATA

As well known the initial data problem for RT spacetimes is within the class of characteristic initial value formulation as opposed to the 1+3 formulation, according to the classification of York[14]. For RT spacetimes the function $K(u_0, \theta, \phi)$ given in a characteristic surface $u = u_0$ corresponds to the initial data to be evolved via the RT equation (4). Our task is now to exhibit an initial $K(u_0, \theta, \phi)$ that represents instantaneously the general collision of two Schwarzschild black holes (head-on collision or not), by extending a procedure outlined in Refs. [2, 15] to construct initial data for RT dynamics in the non-axial case.

In analogy to bispherical coordinates[16] in the 3-dim Cartesian plane Σ , let us introduce the following parametrization for Cartesian coordinates

$$\begin{aligned} x &= \frac{a \sin \theta \sinh \eta}{\cosh \eta + \cos \theta \sinh \eta} \cos \phi, \\ y &= \frac{a \sin \theta \sinh \eta}{\cosh \eta + \cos \theta \sinh \eta} \sin \phi, \\ z &= \pm \frac{a}{\cosh \eta + \cos \theta \sinh \eta}, \end{aligned} \quad (13)$$

for $z > 0$ and $z < 0$ respectively. In the above $0 \leq \eta \leq \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. In this parametrization, $\eta = \eta_0$ corresponds to two spheres, one at $z > 0$ and the other at $z < 0$, centered at $(x = y = 0, z = \pm a \cosh \eta_0)$ respectively, with radius $a \sinh \eta_0$. The Cartesian vector from a point $P : (x, y, z)$ of Σ has length

$$r(\eta, \theta) = a \sqrt{\frac{\cosh \eta - \cos \theta \sinh \eta}{\cosh \eta + \cos \theta \sinh \eta}} \quad (14)$$

For $\eta = \infty$ the spheres degenerate into the planes $z = 0$ and $z = \pm \infty$. The usefulness of this parametrization will become clear in what follows. We note that the Cartesian coordinates are continuous functions, with continuous derivatives, of (η, θ, ϕ) . Singularities occurring are

the usual singularities of a spherical coordinate system. For future reference let us introduce the functions

$$\mathcal{S}_{(\pm)}(\eta, \theta, \phi, \mathbf{n}) = \sqrt{\cosh \eta \pm (\mathbf{n} \cdot \hat{\mathbf{x}}) \sinh \eta}. \quad (15)$$

where $\hat{\mathbf{x}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ and $\mathbf{n} = (n_1, n_2, n_3)$, with $n_1^2 + n_2^2 + n_3^2 = 1$. In the above parametrization (13), the flat space line element $ds^2 =$

$(dx)^2 + (dy)^2 + (dz)^2$ is expressed as

$$ds_{\text{flat}}^2 = \frac{a^2}{\mathcal{S}_{(+)}^4(\eta, \theta, \phi, \mathbf{n})} \left[d\eta^2 + \sinh^2 \eta (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (16)$$

We now take Σ as a spacelike surface of initial data, with geometry defined by the line element

$$ds^2 = a^2 K^2(\eta + \gamma_0, \theta, \phi) \left[d\eta^2 + \sinh^2(\eta + \gamma_0)(d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (17)$$

where γ_0 is an arbitrary parameter. By assuming time-symmetric data (namely, Σ a maximal slice with zero extrinsic curvature) we obtain that the Hamiltonian constraints reduce to ${}^{(3)}R = 0$. With the substitution $K \equiv \Phi^2$, the constraint equation reduces to the Laplace equation

$$\frac{1}{\sin \theta} \left(\Phi_{\theta} \sin \theta \right)_{\theta} + \left(\Phi' \sinh^2(\eta + \gamma_0) \right)' + \frac{1}{\sin^2 \theta} \Phi_{\phi\phi} + \frac{3}{4} \sinh^2(\eta + \gamma_0) \Phi = 0, \quad (18)$$

where a prime denotes derivative with respect to η . It is not difficult to verify that the functions

$$\Phi = \left(\frac{1}{\mathcal{S}_{(\pm)}(\eta + \gamma_0, \theta, \phi, \mathbf{n})} \right) \quad (19)$$

satisfy Eq. (18) and, with respect to metric (17), correspond to flat space solutions (zero curvature). It then follows that

$$\Phi = \left(\frac{\alpha_1}{\mathcal{S}_{(-)}(\eta + \gamma_0, \theta, \phi, \mathbf{n})} \right) + \left(\frac{\alpha_2}{\mathcal{S}_{(+)}(\eta + \gamma_0, \theta, \phi, \tilde{\mathbf{n}})} \right) \quad (20)$$

is a nonflat solution of (18), where α_1 and α_2 are arbitrary positive constants. The nonflat 3-dim geometry defined by (20),

$$ds^2 = a^2 \Phi^4 \left[d\eta^2 + \sinh^2(\eta + \gamma_0)(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (21)$$

is asymptotically flat with a form analogous to that of the 3-dim spatial section of the Schwarzschild geometry in isotropic coordinates, as we proceed to show.

Without loss of generality we take in (20) $\tilde{\mathbf{n}} = (0, 0, 1)$, that corresponds to choose the z -axis along $\tilde{\mathbf{n}}$. In this instance a straightforward manipulation shows that, for $\eta \gg \gamma_0$, the metric (21) can be rewritten as

$$ds^2 = \left(\alpha_2 + \frac{a\alpha_1}{r(\eta, \theta)} \frac{\sqrt{\cosh \eta - \cos \theta \sinh \eta}}{\sqrt{\cosh \eta - (\mathbf{n} \cdot \hat{\mathbf{x}}) \sinh \eta}} \right)^4 ds_{\text{flat}}^2. \quad (22)$$

Now to probe the asymptotic structure of the metric (22) let us consider η very large and, for this η , points (x, y, z) whose distance from the origin is also very large,

namely, when $(\eta \rightarrow \infty, \theta \simeq \pi)$. In this asymptotic limit, returning to Cartesian coordinates, the 3-geometry (22) can be given in the approximate form

$$g_{ij} \simeq \left\{ 1 + \frac{2 M_{(1)}}{r(\eta, \theta)} \right\} \delta_{ij}, \quad (23)$$

where we fixed the scale of bispherical-type coordinates by taking $2\sqrt{2}a = m_0(\alpha_1 + \alpha_2)\sqrt{1 + n_3} \alpha_2/\alpha_1$. The Schwarzschild mass $M_{(1)} = m_0(\alpha_1 + \alpha_2)$.

From the above construction we can now extract initial data for the RT dynamics, which has its initial value problem on null cones. Based on the initial data formulation on characteristic surfaces proposed by D'Inverno and Stachel[17, 18] – in which the degrees of freedom of the vacuum gravitational field are contained in the conformal structure of 2-spheres embedded in a 3-spacelike surface – we are then led to adopt the conformal structure given by the conformal factor (20) defined on the surface $\eta = 0$,

$$K(u_0, \theta, \phi) = \left(\frac{\alpha_1}{\mathcal{S}_{(-)}(\gamma_0, \theta, \phi, \mathbf{n})} + \frac{\alpha_2}{\mathcal{S}_{(+)}(\gamma_0, \theta, \phi, \tilde{\mathbf{n}})} \right)^2, \quad (24)$$

with $\tilde{\mathbf{n}} = (0, 0, 1)$, as initial data for the RT dynamics. This conformal structure is to be extended along null bicharacteristics and propagated along a timelike congruence of the spacetime via RT dynamics. A restricted spacetime may then be constructed locally as the product of the two-sphere geometry times a timelike plane (u, \tilde{r}) generated by a null vector $\partial/\partial \tilde{r}$ and a timelike vector $\partial/\partial u$ with geometry $d\sigma^2 = \alpha^2(u, \tilde{r}, \theta, \phi) du^2 + 2du d\tilde{r}$. The four geometry is then taken as

$$ds^2 = \alpha^2(u, \tilde{r}, \theta, \phi) du^2 + 2du d\tilde{r} - \tilde{r}^2 K^2(u, \theta, \phi) (d\theta^2 + \sin^2 \theta d\phi^2). \quad (25)$$

Eq. (25) is the RT metric, the dynamics of which (ruled by Einstein's vacuum field equations) propagates the initial data (24) forward in time from the characteristic initial surface $u = u_0$. We note that Einstein's vacuum equations demand that the function $\alpha^2(u, \tilde{r}, \theta, \phi)$ has the form given in (2).

The interpretation of the asymptotically flat initial data (24) as two instantaneously interacting

Schwarzschild black holes boosted along the z -axis is now discussed, based on perturbations of the RT metric (25) constructed with such data. As shown in Section II, for $\alpha_1 = 0$ the data correspond in (25) to a static Schwarzschild black hole (with the total Bondi mass $m_0(\alpha_2)^6 \cosh \gamma_0$) boosted along the direction defined by the unit vector $\tilde{\mathbf{n}}$, with $v = \tanh \gamma_0$. For $\alpha_1 \neq 0$, with $\alpha_1 \ll \alpha_2$, the configuration is no longer static and cannot therefore be a black hole, but can still be interpreted as an initially perturbed boosted Schwarzschild black hole. Conversely the same consideration holds for ($\alpha_1 \neq 0, \alpha_2 = 0$) and $\alpha_1 \neq 0$ with $\alpha_2 \ll \alpha_1$, the latter case corresponding also to an initially perturbed boosted

Schwarzschild black hole. In this sense we associate the perturbation with a black hole of relative small rest mass, boosted along the direction \mathbf{n} (or $\tilde{\mathbf{n}}$) in non head-on collision with a larger black hole boosted along the direction $\tilde{\mathbf{n}}$ (or \mathbf{n}). The initial infalling velocity of each black hole considered individually is given by $v = \tanh \gamma_0$.

Without loss of generality, in the remaining of the paper we fix $\alpha_2 = 1$ and drop subscripts 1 and 0 of the parameters α_1 and γ_0 , respectively, in order to avoid over-cluttering in formulae and Figures. In this instance, the initial data (24) will in principle contain four independent parameters, namely, $(\alpha, \gamma, \mathbf{n})$ with $n_1^2 + n_2^2 + n_3^2 = 1$, and assumes the form

$$K(u_0, \theta, \phi) = \left(\frac{1}{\sqrt{\cosh \gamma + \cos \theta \sinh \gamma}} + \frac{\alpha}{\sqrt{\cosh \gamma - (\mathbf{n} \cdot \hat{\mathbf{x}}) \sinh \gamma}} \right)^2. \quad (26)$$

We must comment that, since only two black holes are involved in the collision, the data (26) should actually depend (besides α and γ) only on the incidence angle ρ_0 between the two initial black holes, determined by the scalar product $\mathbf{n}_z \cdot \mathbf{n}$. Therefore we should expect that (i) the resulting dynamics of the system would not be altered under a rigid rotation of the two initial black holes about the z -axis, which could be used to locate \mathbf{n} of the initial data in the $x - z$ plane; and consequently (ii) the initial data will result in a planar dynamics of the collision, a fact that would save a lot of computational effort. The above statements (i)-(ii) will indeed be confirmed by the numerical results obtained from the evolution of the data (26), as shown in section VI, and establish the planar nature of a general non head-on collision of two black holes.

Finally we should remark that, in the full Bondi-Sachs problem, the analysis of field equations in the $2 + 2$ formulation [7, 17] shows that the two *news* functions $c_u^{(1)}(u, \theta, \phi)$ and $c_u^{(2)}(u, \theta, \phi)$ are part of the initial data to be prescribed for the evolution of the system [19]. However for the RT dynamics the *news* are already specified once the initial data for the RT equation, namely $K(u_0, \theta, \phi)$, is given and consequently $K(u, \theta, \phi)$ is given for all $u > u_0$ from the numerical evolution of the data.

The initial data (26) will be evolved numerically – via RT dynamics – up to a final configuration that corresponds to a remnant Schwarzschild black hole boosted along the axis determined by the unit vector \mathbf{n}_f and having the form of the solution (11), as we discuss in the next Section. A numerical code using the Galerkin spectral method is was implemented to integrate the non-axisymmetric RT equation (4), the basis of which is described in the next Section.

IV. NUMERICAL EVOLUTION OF THE DATA

We proceed now to discuss the numerical evolution of the initial data (26) via the non-axisymmetric RT equation (4). Throughout the present Section the variable θ will be expressed in terms of the variable $x = \cos \theta$. The numerical integration of the non-axisymmetric RT equation is performed using a Galerkin spectral method which is now described in detail. In the present procedure we rewrite the equations (3)-(4) using the variable $P(u, x, \phi) \equiv 1/K(u, x, \phi)$ (instead of the former $K(u, x, \phi)$), an approach already adopted in Ref. [20]. We obtain that

$$\lambda(u, x, \phi) = (1 - x^2) [PP_{xx} - P_x^2] - 2xPP_x + P^2 + \frac{1}{(1 - x^2)} [PP_{\phi\phi} - P_\phi^2] \quad (27)$$

and

$$\dot{P}(u, x, \phi) = -\frac{P^3}{12m_0} \left[(1 - x^2)\lambda_{xx} - 2x\lambda_x + \frac{\lambda_{\phi\phi}}{(1 - x^2)} \right]. \quad (28)$$

where a dot and the the subscripts x and ϕ denote, respectively, derivatives with respect to u , x and ϕ .

The Galerkin method establishes that $P(u, x, \phi)$ can be expanded in a convenient set of basis functions of a projection space by which we can reduce the RT partial differential equation (28) into a finite set of nonlinear coupled ordinary differential equations. This set of equations constitutes an autonomous dynamical system, the dimension of which depends directly on the truncation in the Galerkin method to approximate the RT dynamics. As the conformal function $P(u, x, \phi)$ defined on the 2-sphere is assumed to be sufficiently smooth, we can use the real Spherical Harmonics (SHs) [21] as the appropriate basis that better approximates our desired solution. Before starting with the numerical scheme, we first present some definitions and properties of these functions.

Unlike the complex SHs, the real SHs consist of a break of the complex SHs into their sine and cosine parts as follows,

$$\begin{aligned}\langle x, \phi | l, m \rangle^{(+)} &\equiv {}^{(+)}Y_l^m(x, \phi) \equiv W_{l,m} P_l^m(x) \cos(m\phi) \\ &= {}^{(+)}\langle l, m | x, \phi \rangle, \\ \langle x, \phi | l, m \rangle^{(-)} &\equiv {}^{(-)}Y_l^m(x, \phi) \equiv W_{l,m} P_l^m(x) \sin(m\phi) \\ &= {}^{(-)}\langle l, m | x, \phi \rangle,\end{aligned}\quad (29)$$

where $P_l^m(x)$ are the associated Legendre functions defined by

$$P_l^m(x) \equiv (1-x^2)^{|m|/2} \frac{d^m P_l}{dx^m}(x), \quad (30)$$

and the normalization factors $W_{l,m}$ are given by

$$W_{l,m} \equiv \sqrt{\frac{1}{2} \frac{(2l+1)(l-m)!}{\pi(l+m)!}}. \quad (31)$$

From these formulas we see that the case $m = 0$ gives us the basis functions for the axial case, the Legendre polynomials $P_l(x) \equiv \langle x | l \rangle$ [21].

The real SH orthogonality relations are given by

$$\begin{aligned}{}^{(\pm)}\langle l, m | l', m' \rangle^{(\pm)} &\equiv \\ &\equiv \int_0^{2\pi} \int_{-1}^1 {}^{(\pm)}\langle l, m | x, \phi \rangle \langle x, \phi | l', m' \rangle^{(\pm)} dx d\phi \\ &= \int_0^{2\pi} \int_{-1}^1 {}^{(\pm)}Y_l^m(x, \phi) {}^{(\pm)}Y_{l'}^{m'}(x, \phi) dx d\phi \\ &= \delta_{l,l'} \delta_{m,m'},\end{aligned}\quad (32)$$

and

$$\begin{aligned}{}^{(\pm)}\langle l, m | l', m' \rangle^{(\mp)} &\equiv \\ &= \int_0^{2\pi} \int_{-1}^1 {}^{(\pm)}\langle l, m | x, \phi \rangle \langle x, \phi | l', m' \rangle^{(\mp)} dx d\phi \\ &= \int_0^{2\pi} \int_{-1}^1 {}^{(\pm)}Y_l^m(x, \phi) {}^{(\mp)}Y_{l'}^{m'}(x, \phi) dx d\phi = 0,\end{aligned}\quad (33)$$

and are fundamental in the treatment of the Galerkin method. Here the following completeness relation is also used,

$$\int_0^{2\pi} \int_{-1}^1 |x, \phi \rangle \langle x, \phi| dx d\phi = 1. \quad (34)$$

Now we are able to construct our numerical Galerkin scheme, that will allow us to obtain an accurate approximated numerical solution for (28) corresponding to given initial data. As every twice continuously differentiable, suitably periodic real function defined on the surface of a sphere admits an absolutely convergent expansion in terms of the SHs, let us consider the expansion

$$\begin{aligned}P_{(a)}(u, x, \phi) &= \sum_{l=0}^{N_P} \frac{1}{2} A_{l,0}(u) {}^{(+)}Y_l^0(x, \phi) + \\ &+ \sum_{l=1}^{N_P} \left[\sum_{m=1}^l A_{l,m}(u) {}^{(+)}Y_l^m(x, \phi) + \right. \\ &\quad \left. + \sum_{m=1}^l B_{l,m}(u) {}^{(-)}Y_l^m(x, \phi) \right],\end{aligned}\quad (35)$$

as our approximated solution for $P(u, x, \phi)$. Here N_P is a positive integer that defines the truncation order of the Galerkin method. By using the orthogonality relations (32) and (33), the modal coefficients $A_{l,m}(u)$ and $B_{l,m}(u)$ are given by

$$\begin{aligned}A_{l,m}(u) &= \int_0^{2\pi} \int_{-1}^1 {}^{(+)}Y_l^m(x, \phi) P_{(a)}(u, x, \phi) dx d\phi \\ &\equiv {}^{(+)}\langle l, m | P_{(a)} \rangle,\end{aligned}\quad (36)$$

$$\begin{aligned}B_{l,m}(u) &= \int_0^{2\pi} \int_{-1}^1 {}^{(-)}Y_l^m(x, \phi) P_{(a)}(u, x, \phi) dx d\phi \\ &\equiv {}^{(-)}\langle l, m | P_{(a)} \rangle.\end{aligned}\quad (37)$$

Inserting the expanded solution (35) in the RT partial differential equation (28) we obtain an autonomous dynamical system of dimension $(N_P + 1)^2$ for the modal coefficients $(A_{l,m}(u), B_{l,m}(u))$, namely,

$$\begin{aligned}\dot{A}_{l,m}(u) &= \mathcal{A}_{l,m}(A_{l,m}(u), B_{l,m}(u)), \\ \dot{B}_{l,m}(u) &= \mathcal{B}_{l,m}(A_{l,m}(u), B_{l,m}(u)),\end{aligned}\quad (38)$$

where $\mathcal{A}_{l,m}$ and $\mathcal{B}_{l,m}$ are polynomials of order $(N_P + 1)^2$ in $A_{l,m}(u)$ and $B_{l,m}(u)$ and a dot denotes here u -derivative. The Galerkin scheme guarantees that the projections of $(P(u, x, \phi) - P_{(a)}(u, x, \phi))$ onto each basis function, namely, $\langle (P(u, x, \phi) - P_{(a)}(u, x, \phi)), Y_l^m(x, \phi) \rangle$ approach zero when $N_P \rightarrow \infty$ so that (35) approaches the exact solution $P(u, x, \phi)$, in the sense of the norm of the projection basis space of the SHs. The initial conditions $(A_{l,m}(u_0), B_{l,m}(u_0))$ to be used to integrate the dynamical system (38) are provided by the initial data $K(u_0, x, \phi)$ constructed in last Section (Eqs. (26)). These initial values are obtained from the Galerkin decomposition of $P(u_0, x, \phi)$ (cf. Eq. (35)) evaluated according to (36).

As a matter of fact, the projections that transform the RT equation into the dynamical system (38) demands actually that we replace $P(u, x, \phi)$ by $P_{(a)}(u, x, \phi)$ in (27) and (28). But in doing so we excessively increase the computational demand, overloading the computer data storage. To circumvent this problem, we divide the RT equation into two parts that we called constraints: the λ -constraint and the P^3 -constraint. The process will consist then in also expanding – in the same way as in (35) – the expressions for λ and P^3 ,

$$\begin{aligned}\lambda_{(a)}(u, x, \phi) &= \sum_{l=0}^{N_\lambda} \frac{1}{2} E_{l,0}(u) {}^{(+)}Y_l^0(x, \phi) + \\ &+ \sum_{l=1}^{N_\lambda} \sum_{m=1}^l E_{l,m}(u) {}^{(+)}Y_l^m(x, \phi) + \\ &+ \sum_{l=1}^{N_\lambda} \sum_{m=1}^l F_{l,m}(u) {}^{(-)}Y_l^m(x, \phi),\end{aligned}\quad (39)$$

$$P^3_{(a)}(u, x, \phi) = \sum_{l=0}^{N_{P^3}} \frac{1}{2} C_{l,0}(u) {}^{(+)}Y_l^0(x, \phi) +$$

$$\begin{aligned}
& + \sum_{l=1}^{N_{P3}} \sum_{m=1}^l C_{l,m}(u)^{(+)} Y_l^m(x, \phi) + \\
& + \sum_{l=1}^{N_{P3}} \sum_{m=1}^l D_{l,m}(u)^{(-)} Y_l^m(x, \phi). \quad (40)
\end{aligned}$$

Here, N_λ and N_{P3} are integers that define the truncation orders for the respective expansions. Throughout the paper, we take $N_P = N_\lambda = N_{P3}$, but these orders are totally independent and can be taken with different values. Now, we have a new two set of modal coefficients ($\{C_{l,m}, D_{l,m}\}$ and $\{E_{l,m}, F_{l,m}\}$) that will be uniquely determined by the first employed set of modal coefficients $\{A_{l,m}, B_{l,m}\}$ by

$$\begin{aligned}
C_{l,m}(u) & \equiv {}^{(+)} \langle l, m | P_{(a)}^3 \rangle \\
& = C_{l,m}(\{A_{l,m}(u), B_{l,m}(u)\}), \\
D_{l,m}(u) & \equiv {}^{(-)} \langle l, m | P_{(a)}^3 \rangle \\
& = D_{l,m}(\{A_{l,m}(u), B_{l,m}(u)\}), \\
E_{l,m}(u) & \equiv {}^{(+)} \langle l, m | \lambda_{(a)} \rangle \\
& = E_{l,m}(\{A_{l,m}(u), B_{l,m}(u)\}), \\
F_{l,m}(u) & \equiv {}^{(-)} \langle l, m | \lambda_{(a)} \rangle \\
& = F_{l,m}(\{A_{l,m}(u), B_{l,m}(u)\}). \quad (41)
\end{aligned}$$

The RT dynamical equation will have then a new form: the left side will be written by the u -derivative of $P_{(a)}$

$$\begin{aligned}
\dot{P}_{(a)}(u, x, \phi)_{left} & = \sum_{l=0}^{N_P} \frac{1}{2} \dot{A}_{l,0}(u)^{(+)} Y_l^0(x, \phi) + \\
& + \sum_{l=1}^{N_P} \sum_{m=1}^l \dot{A}_{l,m}(u)^{(+)} Y_l^m(x, \phi) + \\
& + \sum_{l=1}^{N_P} \sum_{m=1}^l \dot{B}_{l,m}(u)^{(-)} Y_l^m(x, \phi), \quad (42)
\end{aligned}$$

and, its right side, will be given by the replacement of λ and P^3 in (28) by the constraint expressions (39) and (40).

$$\dot{P}_{(a)}(u, x, \phi)_{right} = -\frac{P_{(a)}^3}{12m_0} \left[\lambda_{(a)xx} - 2x\lambda_{(a)x} + \frac{\lambda_{(a)\phi\phi}}{(1-x^2)} \right].$$

Finally we project both sides to get the relation between the u -variation of the set $\{A_{l,m}(u), B_{l,m}(u)\}$ and the new modal coefficients

$$\begin{aligned}
\dot{A}_{l,m}(u) & \equiv {}^{(+)} \langle l, m | \dot{P}_{(a)} \rangle \\
& = \dot{A}_{l,m}(\{C_{l,m}(u), D_{l,m}(u), E_{l,m}(u), F_{l,m}(u)\}) \\
\dot{B}_{l,m}(u) & \equiv {}^{(-)} \langle l, m | \dot{P}_{(a)} \rangle \\
& = \dot{B}_{l,m}(\{C_{l,m}(u), D_{l,m}(u), E_{l,m}(u), F_{l,m}(u)\}) \quad (43)
\end{aligned}$$

This completes our scheme to obtain the dynamical system version of the non-axisymmetric RT equation (28). To solve it, we use a fourth-order Runge-Kutta recursive method adapted to our constraints. So we have to get the initial values for the modal coefficients, say

$$\{A_{l,m}(u_0), B_{l,m}(u_0)\}$$

$$\begin{aligned}
A_{l,m}(u_0) & \equiv {}^{(+)} \langle l, m | P_{0(a)} \rangle, \\
B_{l,m}(u_0) & \equiv {}^{(-)} \langle l, m | P_{0(a)} \rangle. \quad (44)
\end{aligned}$$

The effect of the truncation N_P on the initial data may be evaluated by the relative error $\mathcal{R}(u_0)$ between the approximate and exact expressions given by (35) and $P(u_0, x, \phi) \equiv 1/K(u_0, x, \phi)$ (cf. (26)),

$$\mathcal{R}(u_0, x) = \frac{|P(u_0, x, \phi) - P_{(a)}(u_0, x, \phi)|}{P(u_0, x, \phi)}. \quad (45)$$

In all the numerical experiments of the paper we have adopted $N_P = 7$; with this truncation the relative error above is of the order of, or smaller than 10^{-8} , for all $-1 \leq x \leq 1$ and $0 \leq \phi \leq 2\pi$. The RT dynamics furnishes us with a further test to check the accuracy and reliability of our numerical codes. In fact, for any sufficiently smooth $K(u, x)$ we have that the quantity

$$\zeta(u) = \int_0^{2\pi} d\phi \int_{-1}^1 P^{-2}(u, x, \phi) dx, \quad (46)$$

is conserved along the dynamics, namely, $\partial_u \zeta(u) = 0$. Evaluating its exact value from the initial data (26) and at distinct steps of the computation we that $|\zeta(u_0) - \zeta(u)| \leq 10^{-10}$ for all our numerical experiments and for all the sampled values of $u > u_0$, for the adopted truncation $N_P = 7$.

Our numerical experiments are realized with the initial data (26) – which corresponds to one initial black hole boosted along the z -axis and the second black hole moving along the direction determined by $\mathbf{n} = (n_1, n_2, n_3)$ with $n_1^2 + n_2^2 + n_3^2 = 1$ – having in principle four independent parameters.

We vary α in the interval $(0, 1.5)$ for $\gamma = 0.2$ and several values of n_1 . Exhaustive numerical experiments show that for a sufficiently large computation time u_f all modal coefficients become constant, namely, at u_f the emission of gravitational waves is considered to have ceased. For u_f we actually have that $|A_{l,m}(u_f + h) - A_{l,m}(u_f)| \leq 10^{-10}$, $|B_{l,m}(u_f + h) - B_{l,m}(u_f)| \leq 10^{-12}$ for all $l = 0 \dots 7$, where h is the integration step.

From these modal coefficients $A_{l,m}(u_f)$ and $B_{l,m}(u_f)$ we reconstruct the final configuration $P(u_f, x, \phi) \simeq P_{(a)}(u_f, x, \phi)$ that, in all cases, can be expressed as

$$\begin{aligned}
P(u_f, x, \phi) & = \frac{A_{0,0}(u_f)}{2} + \frac{A_{1,0}(u_f)}{2} \cos \theta \\
& + A_{1,1}(u_f) \sin \theta \cos \phi \\
& + B_{1,1}(u_f) \sin \theta \cos \phi + \mathcal{O}(10^{-10}) \\
& \simeq \frac{1}{K_f} (\cosh \gamma_f + \mathbf{n}_f \cdot \hat{\mathbf{x}} \sinh \gamma_f). \quad (47)
\end{aligned}$$

The rms error of the second equality in (47) is of the order of, or smaller than 10^{-12} . The expression in the second equality of (47) corresponds to a boosted black hole along the direction $\mathbf{n}_f = (n_{1f}, n_{2f}, n_{3f})$ with boost

parameter γ_f and rest mass $m_0 K_f^3$, a confirmation of the Chrusiel-Singleton theorem[10].

From (47) we can identify

$$\begin{aligned} A_{0,0}(u_f) &= \frac{2 \cosh \gamma_f}{K_f}, \\ A_{1,0}(u_f) &= \frac{2 n_{3f} \sinh \gamma_f}{K_f}, \\ A_{1,1}(u_f) &= \frac{n_{1f} \sinh \gamma_f}{K_f}, \\ B_{1,1}(u_f) &= \frac{n_{2f} \sinh \gamma_f}{K_f}, \end{aligned} \quad (48)$$

up to $\mathcal{O}(10^{-12})$, from which we can read the final parameters of the remnant stationary black hole $(K_f, \gamma_f, \mathbf{n}_f)$. It results

$$\begin{aligned} n_{1f} &= \frac{2A_{1,1}(u_f)}{A_{1,0}(u_f)} n_{3f}, \quad n_{2f} = \frac{2B_{1,1}(u_f)}{A_{1,0}(u_f)} n_{3f} \\ n_{3f} &= \left(1 + \left(\frac{2A_{1,1}(u_f)}{A_{1,0}(u_f)} \right)^2 + \left(\frac{2B_{1,1}(u_f)}{A_{1,0}(u_f)} \right)^2 \right)^{-1/2} \\ \gamma_f &= \tanh^{-1} \left(\frac{1}{n_{3f}} \frac{A_{1,0}}{A_{0,0}} \right) \\ K_f &= \frac{2}{A_{0,0}} \cosh \gamma_f. \end{aligned} \quad (49)$$

An alternative to evaluate the parameter K_f is the use of the initial data in the conserved quantity (46), namely,

$$K_f = \left(\frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^1 K^2(u_0, x, \phi) dx \right)^{1/2}. \quad (51)$$

The agreement is within 10 significant decimal digits. This is also a test of the accuracy of $(n_{1f})^2 + (n_{2f})^2 + (n_{3f})^2 = 1$.

One of the basic results to be extracted from our numerical experiments are the values of (K_f, γ_f, n_{3f}) , for each of the independent parameters $(\alpha, \gamma, \mathbf{n})$ of the initial data. The former are the basic parameters of the remnant that – together with the initial data of the system and the function $K(u, \theta, \phi)$ for all $u_0 \leq u \leq u_f$ – allow us to evaluate quantities which are characteristic of the radiative transfer processes involved in the gravitational wave emission. Finally we remark that the integration of the dynamical system (43) used a fourth-order Runge-Kutta recursive method (adapted to our constraints) together with a C++ integrator. Unless otherwise stated, all our numerical results are restricted to the choice $\gamma = 0.2$, corresponding to the initial infalling collision velocity $v \simeq 0.1973$.

V. THE PLANAR NATURE OF THE GENERAL NON-HEAD-ON COLLISION

From the numerical experiments done to check the long time evolution of the initial data (26) and to obtain the basic parameters of the remnant black hole, we observe that the final direction of remnant velocity is contained in the plane determined by the two directions of the initial individual colliding black holes. For illustration let us consider the evolution of the initial data (26), with $\mathbf{n}_z = (0, 0, 1)$ corresponding to the direction of the initial colliding black hole (boosted along the positive z -axis), with $\mathbf{n} = (0.26, 0.07, 0.963068)$ corresponding to the direction of the second initial colliding black hole and the several values of $\alpha = 0.1, 0.1, 0.20, 0.3, 0.50$. The resulting final direction of the remnant is given by (cf. (49))

$$\begin{aligned} \mathbf{n}_f(\alpha = 0.1) &\simeq (1.363345 \cdot 10^{-2}, 3.670544 \cdot 10^{-3}, 9.999003 \cdot 10^{-1}), \\ \mathbf{n}_f(\alpha = 0.2) &\simeq (3.064877 \cdot 10^{-2}, 8.251592 \cdot 10^{-3}, 9.994961 \cdot 10^{-1}), \\ \mathbf{n}_f(\alpha = 0.3) &\simeq (0.000000 \cdot 10^{-2}, 0.000000 \cdot 10^{-3}, 9.000000 \cdot 10^{-1}), \\ \mathbf{n}_f(\alpha = 0.5) &\simeq (1.197374 \cdot 10^{-1}, 3.223699 \cdot 10^{-2}, 9.922820 \cdot 10^{-1}). \end{aligned}$$

In all cases we obtain

$$\mathbf{n}_z \cdot \mathbf{n} \wedge \mathbf{n}_f \simeq 0. \quad (52)$$

Within machine precision, (52) is valid up to $\mathcal{O}(10^{-10})$, and it also holds for all numerical experiments done in the paper considering a large domain of the initial data parameters, implying that the dynamics is indeed planar, as discussed at the end of Section III. The result (52) could also be considered as an additional test for the accuracy of our numerical code.

This fundamental result – which will be used to save computational efforts in the dynamical system evolution of the RT non-axisymmetric dynamics – has an important physical counterpart connected to the fact that the linear momentum flux carried out by gravitational waves is confined to the plane of the initial collision. In other words, the linear momentum of the merged system in the direction $\mathbf{n}_z \wedge \mathbf{n}$ is conserved, as discussed in Section VIII.

Let us then consider a rigid rotation of the two initial black hole system about the direction of motion of the first initial black, namely, the z -axis. This rotation will be chosen so that the plane of the initial collision, determined by the two directions $\mathbf{n}_z = (0, 0, 1)$ and \mathbf{n} , coincide with the $x - z$ plane. In this case the direction of motion of the second initial colliding black hole is now expressed by the unit vector $\mathbf{n} = (n_1, 0, n_2)$, with $n_1^2 + n_2^2 = 1$. In this instance the initial data (26) can assume the simpler form

$$K(u_0, \theta, \phi) = \left(\frac{1}{\sqrt{\cosh \gamma + \cos \theta \sinh \gamma}} + \frac{\alpha}{\sqrt{\cosh \gamma - (\cos \rho_0 \cos \theta + \sin \rho_0 \sin \theta \cos \phi) \sinh \gamma}} \right)^2. \quad (53)$$

This initial data contains now three independent pa-

rameters (α, γ, ρ_0) only, where ρ_0 is the angle formed by \mathbf{n}

and the z -axis and is used to parametrize $n_1 = \cos \rho_0$ and $n_2 = \sin \rho_0$. Again, as already established, the motion of the resulting remnant black hole will then be restricted to the $x - z$ plane for any values of the initial data parameters. From the preceding discussions of the present Section we have that the initial data (53) describes a general non head-on collision of two black holes in the realm of RT dynamics.

Now, for exhaustive numerical tests with the initial data (53) and a large range of parameters (α, ρ_0) we verify that: (i) the initial modal coefficients $B_{l,m}(u_0) \lesssim 10^{-17}$, and (ii) its evolution via the RT dynamical system (43) maintains $B_{l,m}(u) \lesssim 10^{-17}$, for all (l, m) in their allowed range, and for all $u_0 < u \leq u_f$. This corresponds – within the precision of our computation – to $B_{l,m}(u) = 0$ for all u . Obviously, from the second equation (49), we have that the unit vector determining the direction of the remnant will always have the form $\mathbf{n} = (n_{1f}, 0, n_{3f})$. Therefore our computational task can be simplified by taking all the $B_{l,m}$ coefficients equal to zero, which is equivalent to restrict all our expansions in the Galerkin method decomposition to the cosine series only. In the remaining of the paper we have adopted this procedure in the numerical evaluations of RT dynamics for the data (53). We have however made sample tests by verifying the absolute differences in the physical results originating from data either from the complete decomposition or the decomposition using the cosine series only. The difference remains always of the order of, or smaller than 10^{-17} . The angle ρ_0 ($0 \leq \rho_0 \leq \pi/2$) will be denoted the collision angle, the limiting cases $\rho_0 = 0$ corresponding to a head-on collision and $\rho_0 = \pi/2$ corresponding to a right angle collision.

It will be possible to follow the full evolution of (53) by considering from very small α up to those values for which the nonlinearities start to play an important role in the dynamics. We will restrict ourselves to the range $0 < \alpha \leq 1.2$. The range of the initial incidence angle in the numerical experiments will be $0 \leq \rho_0 \leq 90^\circ$, the limits corresponding respectively to a head-on collision and an orthogonal collision.

VI. THE BONDI-SACHS FOUR MOMENTUM AND CONSERVATION LAWS

RT spacetimes describe the asymptotically flat exterior gravitational field of a bounded system radiating gravitational waves and in this sense they are in the realm of the 2+2 Bondi-Sachs formulation of gravitational waves in General Relativity[7, 8]. Furthermore initial data for RT dynamics are prescribed on null characteristic surfaces. Therefore suitable expressions for the physical quantities to be used in the description of gravitational wave emission processes and its conservation laws must be derived. To exhibit such expressions it is necessary to perform a coordinate transformation from RT coordinates used in (1) to a coordinate system in which the metric coefficients

satisfy the Bondi-Sachs boundary conditions. It should be noticed that in the RT coordinate system the presence of the term $2rK_u/K$ does not fulfill the appropriate boundary conditions. Although the coordinate transformations from RT coordinates to Bondi-Sachs coordinates cannot be expressible in a closed form (they are given by an infinite series in powers of r^{-1}) [22], their asymptotic expansion allows us to obtain the form of the required physical quantities. In this section we will restrict ourselves to the Bondi-Sachs energy-momentum for the RT spacetimes as well as its conservation laws. Our derivation for the non-axisymmetric case[19] follows closely the work of Göna and Kramer[23] done for the axisymmetric case.

From the supplementary vacuum Einstein equations $R_{UU} = 0$, $R_{U\Theta} = 0$, and $R_{U\Phi} = 0$ in the 2 + 2 Bondi-Sachs formulation[7, 8] (where (U, R, Θ, Φ) are the Bondi-Sachs coordinates), we obtain in RT coordinates

$$\frac{\partial m(u, \theta, \phi)}{\partial u} = -K \left(c_u^{(1)2} + c_u^{(2)2} \right) + \frac{1}{2} \frac{\partial}{\partial u} \left[3c_\theta^{(1)} \cot \theta + 4c_\phi^{(2)} \frac{\cos \theta}{\sin^2 \theta} - 2c_\theta^{(1)} + c_{\theta\theta}^{(1)} + \frac{2}{\sin \theta} c_{\theta\phi}^{(2)} - \frac{1}{\sin^2 \theta} c_{\phi\phi}^{(1)} \right] \quad (54)$$

where $m(u, \theta, \phi)$ is the Bondi mass function and $c_u^{(1)}(u, \theta, \phi)$ and $c_u^{(2)}(u, \theta, \phi)$ are the two *news* functions for the non-axisymmetric case[19], corresponding to the two modes of polarization of the gravitational waves. The extra factor K in the first term of the second-hand-side of Eq. (54) comes from the transformation

$$\lim_{r \rightarrow \infty} \frac{\partial U}{\partial u} = \frac{1}{K}, \quad (55)$$

U being the Bondi time coordinate. For the *news* satisfying the appropriate boundary conditions $c^{(1)} = c^{(2)} = 0$ and $c_\theta^{(1)} = c_\theta^{(2)} = 0$ at $\theta = 0, \pi$, we obtain the Bondi-Sachs four-momentum conservation

$$\frac{dP^\mu(u)}{du} = P_W^\mu, \quad (56)$$

where the Bondi-Sachs four-momentum $P^\mu(u)$ is defined as

$$P^\mu(u) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi m(u, \theta, \phi) l^\mu \sin \theta d\theta, \quad (57)$$

and

$$P_W^\mu(u) = -\frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi K l^\mu \left(c_u^{(1)2} + c_u^{(2)2} \right) \sin \theta d\theta, \quad (58)$$

corresponds to the net flux of energy-momentum carried out by the gravitational waves emitted. In the above $l^\mu = (1, -\sin \theta \cos \phi, -\sin \theta \sin \phi, -\cos \theta)$ is a null vector relative to an asymptotic Lorentz frame at infinity. We note that the last term in (54) vanishes in the integrations due to the boundary conditions of the *news*.

For $\mu = 0$, Eq. (56) yields the Bondi mass formula

$$\frac{dM_B(u)}{du} = -P_W(u) \quad (59)$$

where $M_B(u)$ is the Bondi mass at a time u and

$$P_W(u) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi K \left(c_u^{(1)2} + c_u^{(2)2} \right) \sin \theta d\theta. \quad (60)$$

is the power extracted from the system by the gravitational wave emission in a time u . The total energy E_W carried out of the system by the gravitational waves is given by the time integral of (60) up to u_f , and corresponds to the total Bondi mass extracted from the system, $M_B(u_0) - M_B(u_f) = E_W$, where $M_B(u_f) = m_0 K_f^3 \cosh \gamma_f$ is the Bondi mass of the remnant.

For $\mu = x, y, z$, Eqs. (56) determine the linear momentum conservation of the system,

$$\frac{d \mathbf{P}(u)}{du} = \mathbf{P}_W(u) \quad (61)$$

where the vector

$$\mathbf{P}_W(u) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi K \hat{\mathbf{x}} \left(c_u^{(1)2} + c_u^{(2)2} \right) \sin \theta d\theta. \quad (62)$$

is the net momentum flux carried out by the gravitational waves emitted. In the above $\hat{\mathbf{x}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$.

We are now able to analyze the radiative processes that lead the merged system from its initial configuration to the final configuration of the remnant black hole.

VII. THE POWER EXTRACTED BY THE GRAVITATIONAL WAVE EMISSION AND THE EFFICIENCY OF THE MASS-ENERGY RADIATIVE TRANSFER: UPPER AND LOWER BOUNDS

We now examine the energy extracted from the merged system by the gravitational waves. From our discussion in Section VI on the Bondi-Sachs conservation laws, and specifically from Eqs. (59) and (60) we have that the power emitted by the system in a time u is given by

$$\begin{aligned} P_W(u) &\equiv dE_W(u)/du \\ &= \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi K \left(c_u^{(1)2} + c_u^{(2)2} \right) \sin \theta d\theta. \end{aligned} \quad (63)$$

In Figs. 1 and 2 we plot the power emitted $P_W(u)$ as a function of u/m_0 for $\alpha = 0.1$ and $\alpha = 0.6$, respectively, and $\gamma = 0.2$ fixed, and for several values of the incidence angle ρ_0 . The total energy emission $E_W(u_f)$ in each case is measured by the area below the respective curve – in accordance with (60). Two important features are to be noted in the Figures. First, we can see that a head-on collision ($\rho_0 = 0$) constitutes an upper-bound for the total energy emitted, as well as the orthogonal collision ($\rho_0 = 90^\circ$) constitutes a lower-bound for this energy. This pattern is typical for any $0 < \alpha < 1$ and for all γ . Actually in our numerical experiments we verified this important result for $0 \leq \alpha \leq 1$, consistent with the curves of the efficiency of mass-energy extraction by

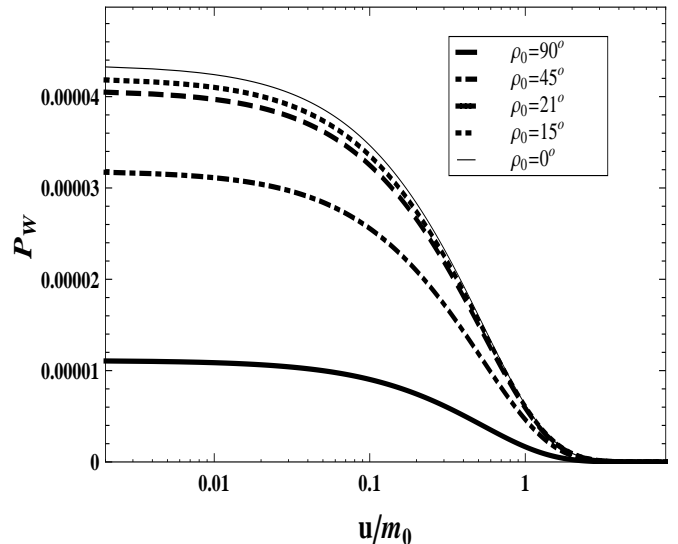


FIG. 1: Linear-log plot of the energy flux (power) carried out by gravitational waves as a function of u/m_0 , for $\alpha = 0.1$ and $\gamma = 0.2$ and several values of the collision angle ρ_0 . We see that, for this relatively small value of α , the gw emission corresponds to a pulse of short duration, $\Delta u/m_0 \sim 3.5$, and its initial intensity decreases as ρ_0 increases. The case $\rho_0 = 0$ (head-on collision) constitutes an upper bound, while the orthogonal collision ($\rho_0 = 90^\circ$) constitutes a lower bound for the total energy emitted (the area below the curve), in accordance with the efficiency behavior.

gravitational waves, as we will discuss later in the present Section.

Second, although the curves present the same behavior for distinct α 's, we can see that for α small (cf. Fig. 1 for $\alpha = 0.1$) the emission corresponds to a short pulse of gravitational waves. We indeed obtain that, for any of the incident angles ρ_0 (here including the head-on collision), the initial power emitted decreases by three orders of magnitude in an interval of time $\Delta u/m_0 \sim 3.5$. On the contrary, for $\alpha = 0.6$ (cf. Fig. 2) the initial power emitted (which is one order of magnitude smaller than in the case $\alpha = 0.1$) decreases by three orders of magnitude in a much larger interval $\Delta u/m_0 \sim 72$ for all the incidence angles ρ_0 considered. In fact, analogous to the case of head-on collision analyzed in Ref. [2], there is a threshold value of $\alpha \sim 0.67$ that separates regimes of short bursts of gravitational waves from a regime of quiescent long time emission.

For α in the interval $(0, 1.5]$ and for several values of ρ_0 in the interval $(0, \pi/2)$ we have determined (K_f, γ_f, ρ_f) which characterize the final boosted black hole configuration. From these parameters of the black hole remnant and the total radiated energy $E_W(u_f)$, the efficiency Δ of mass-energy extraction by gravitational wave emission can be evaluated. According to Eardley[24] the efficiency

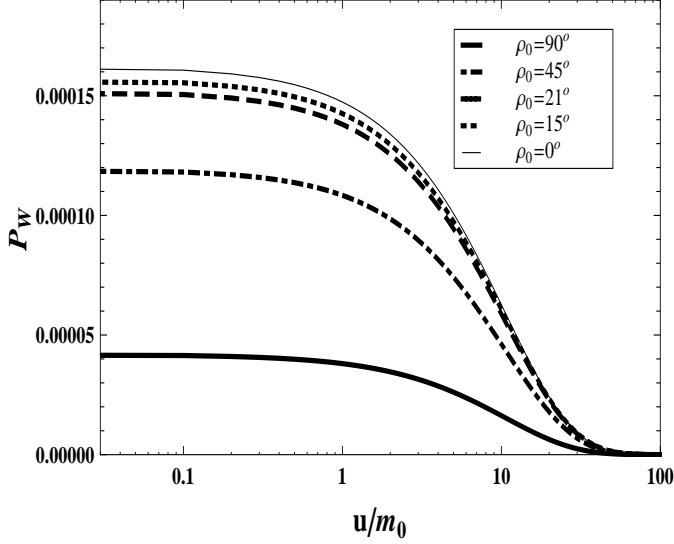


FIG. 2: Linear-log plot of the energy flux (power) carried out by gravitational waves as a function of u/m_0 , for $\alpha = 0.6$ and $\gamma = 0.2$ and several values of the collision angle ρ_0 . Analogous to the case of $\alpha = 0.1$, the head-on collision constitutes an upper bound, and the orthogonal collision a lower bound for the total energy emitted in accordance with the efficiency behavior. The total energy emitted is spread on a larger interval of time, $\Delta u/m_0 \sim 72$.

Δ is defined as

$$\Delta \equiv \frac{(M_B(0) - M_B(u_f))}{M_B(0)}, \quad (64)$$

where $M_B(u_0)$ is the initial Bondi mass and $M_B(u_f)$ is the final Bondi mass. We can alternatively express Δ as

$$\Delta = \frac{E_W(u_f)}{m_0 K_f^3 \cosh \gamma_f + E_W(u_f)}. \quad (65)$$

In Fig.1 we display the log-linear plot of the efficiency Δ versus the mass ratio parameter α for $\gamma = 0.2$ and several values of the collision angle ρ_0 varying between $\rho_0 = 0$ (head-on collision) and $\rho_0 = 90^\circ$ (orthogonal collision). We see that the case of head-on collisions constitute an upper bound for the efficiency Δ while the case of orthogonal collisions constitute a lower bound for Δ , in accordance with the power extraction behavior discussed above and illustrated in Figs. 3 and 4. The efficiencies also decrease as the initial angle of collision ρ_0 increases.

We can also derive that, due to the form of the initial data (53), the efficiency Δ has a maximum for $\alpha = 1$. In fact, let us consider $\alpha > 1$ in (53) and factorize α so that the initial data will turn out to be the product of α^2 times a new $K(u_0, \theta, \phi)$, the latter corresponding to the initial data of two colliding black holes, one with mass parameter $(1/\alpha)$ boosted along the negative z axis and the other with mass parameters 1 boosted along the direction $\mathbf{n} = (\sin \rho_0, 0, \cos \rho_0)$. By a rigid

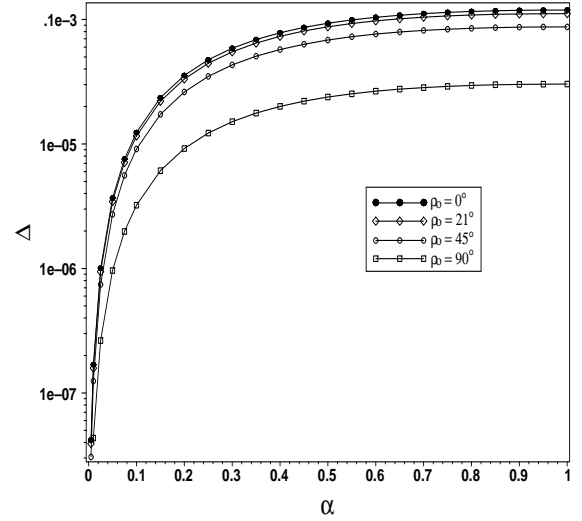


FIG. 3: Log-linear plot of the efficiency Δ for the range $\alpha_1 = (0, 1]$ and several values of the incidence angle ρ_0 . The initial parameter $\gamma = 0.2$ is fixed but the distribution of points for other values of γ exhibit an analogous behavior. The points are connected for a better visualization. Values of Δ are redundant due to the relation $\Delta(\alpha) = \Delta(1/\alpha)$ for $\alpha \in (0, \infty)$.

counter clockwise rotation of the $x - z$ plane, about the Cartesian axis y , of an angle $\pi - \rho_0$ the unity vectors \mathbf{n} and $\tilde{\mathbf{n}} = (0, 0, 1)$ will transform respectively into $\mathbf{n} = (0, 0, -1)$ and $\tilde{\mathbf{n}} = (-\sin \rho_0, 0, -\cos \rho_0)$ reproducing (53) for $\alpha \rightarrow 1/\alpha$. Now it is not difficult to see that, under the rescale of $K(u_0, \theta, \phi)$ by a factor α^2 , both $M_B(u_f)$ and $E_W(u_f)$ rescale with α^6 so that the efficiency Δ satisfies

$$\Delta(\alpha) = \Delta(1/\alpha), \quad (66)$$

for $\alpha \in (0, \infty)$. The rescale of E_W with α^6 uses Eq. (60) and also that du rescales with α^8 , the latter derived directly from the RT equation.

Since we can see numerically that Δ increases with α in the range $\alpha = (0, 1)$ it necessarily follows that Δ has a maximum at $\alpha = 1$ for any $0 < \rho_0 \leq 90^\circ$, as illustrated in Fig. 4. This property holds also for any γ . Also we have that the difference of the efficiencies between the limit configurations, $\rho_0 = 0^\circ$ (upper bound) and $\rho_0 = 90^\circ$ (lower bound), increase as α increases, reaching a maximum at $\alpha = 1$. For the present case $\gamma = 0.2$ we obtain numerically that $\Delta_{\max}(\rho_0 = 0^\circ) \simeq 1.191990 \times 10^{-4}$, $\Delta_{\max}(\rho_0 = 21^\circ) \simeq 1.115352 \times 10^{-4}$, $\Delta_{\max}(\rho_0 = 45^\circ) \simeq 0.872631 \times 10^{-4}$, $\Delta_{\max}(\rho_0 = 90^\circ) \simeq 3.029724 \times 10^{-5}$.

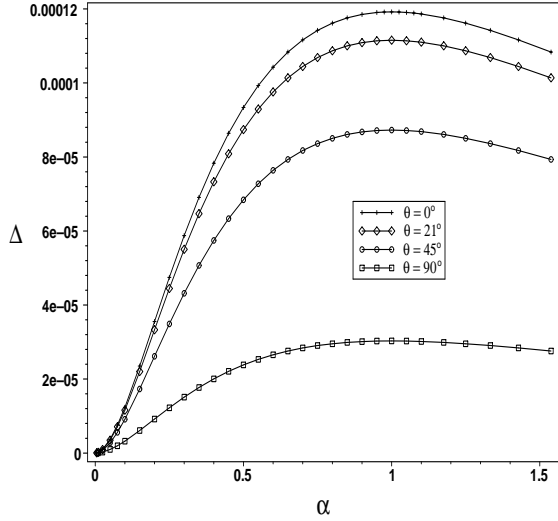


FIG. 4: Plot of the efficiency Δ in the range $\alpha_1 = (0, 1.55)$ exhibiting the absolute maxima at $\alpha = 1$ for $\rho_0 = 90^\circ$, 45° , 21° and 0° . The points are connected for a better visualization. This behavior of maxima at $\alpha = 1$ actually holds for any $0 < \rho_0 \leq 90^\circ$ and any γ .

VIII. GRAVITATIONAL WAVE RECOIL: THE MOMENTUM FLUX CARRIED OUT BY GRAVITATIONAL WAVES AND THE MOMENTUM OF THE REMNANT

The analysis of the linear momentum extraction from the merged system by the gravitational waves is made using the Bondi-Sachs conservation laws (56) for $\mu = x, y, z$, which reads

$$\mathbf{P}_W(u) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi K \hat{\mathbf{x}} \left(c_u^{(1)2} + c_u^{(2)2} \right) \sin\theta d\theta, \quad (67)$$

where $\mathbf{P}_W(u)$ is the momentum flux carried out by the gravitational waves emitted. In accordance with the results and discussions of Section VI the linear momentum of the merged system orthogonal to the plane of collision is conserved, which in our choice of data (53) implies that $P_W^y = 0$, namely, the net flux of linear momentum carried out by gravitational waves is restricted to the plane of collision. Therefore from (67) we have

$$P_W^x(u) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi \sin^2\theta \cos\phi K \left(c_u^{(1)2} + c_u^{(2)2} \right) d\theta, \quad (68)$$

$$P_W^z(u) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi \cos\theta \sin\theta K \left(c_u^{(1)2} + c_u^{(2)2} \right) d\theta, \quad (69)$$

as the relevant momentum fluxes contributing in the conservation law (56) and determining the total impulse imparted on the merged system by gravitational wave emission.

The total momentum conservation and the issue of kicks on the merged system were already analyzed in [3] for the case of a head-on collision ($\rho_0 = 0^\circ$). For values

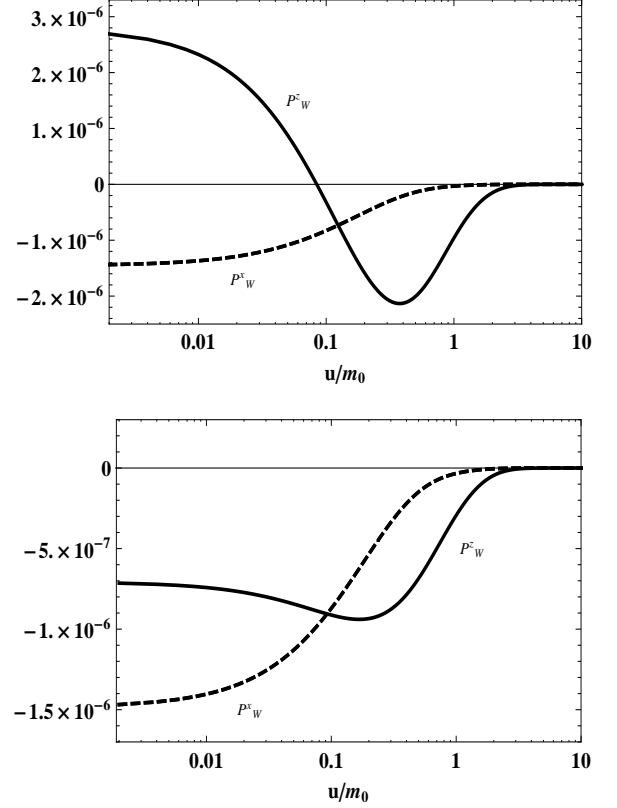


FIG. 5: Linear-log plots of the net momentum fluxes P_W^z and P_W^x carried out by the gravitational waves, for $\alpha = 0.1$, $\gamma = 0.2$, and incidence angles $\rho_0 = 15^\circ$ (top) and $\rho_0 = 90^\circ$ (bottom). Due to the planar nature of the collision, the momentum flux P_W^y is conserved. For the case of the relatively small incidence angle $\rho_0 = 15^\circ$ (top) we have an initial regime with positive $P_W^z(u)$, $0 < u < u_k$, and a consequent increase of the linear momentum z -component of the merged system, $dP^z(u)/du > 0$ (cf. (67)). Along the x axis the merged system is decelerated for all u until the final configuration is reached. For a large value of $\rho_0 = 90^\circ$ (bottom) the pattern changes and the system is decelerated along both directions x and z . The initial phase with $P_W^z(u)$ is not present for $\rho_0 \gtrsim 61^\circ$, as shown in Fig. 6.

of the incidence angle ρ_0 sufficiently small the picture is analogous to that of a head-on collision, with $P_W^x(u) \simeq 0$ for all $u_0 < u < u_f$. However as ρ_0 increases a negative net momentum flux appears along the x -direction leading to a decrease of the momentum of the merged system in that direction for all $0 < u < u_f$. This is illustrated in Fig. 5 (top) where the net momentum fluxes P_W^z and P_W^x carried out by the gravitational waves are plotted as a function of u , for mass ratio parameter $\alpha = 0.1$ and incidence angle $\rho_0 = 15^\circ$. We see that we have an initial phase where the momentum of the merged system along the z axis increases, due to a positive net momentum flux $P_W^z(u)$ for $u_0 < u < u_k$, where $u_k/m_0 \simeq 0.085$ is the time when P_W^z changes sign. Therefore we have a

dominant deceleration regime of the merged system until the remnant black hole configuration is reached. For $\alpha = 0.1$ fixed, this basic pattern is maintained up to $\rho_0 \simeq 61^\circ$, beyond which both net gravitational wave momentum fluxes, $P_W^x(u)$ and $P_W^z(u)$, are negative for the whole domain of evolution $u_0 \leq u \leq u_f$. Actually for $\rho_0 \gtrsim 61^\circ$ the initial positive net momentum flux along z disappears, with the system being decelerated along both directions x and z in the whole domain of the dynamical evolution. This is illustrated in Fig. 5 (bottom) for the limiting case $\rho_0 = 90^\circ$ (orthogonal collision). The behavior of the threshold configuration $\rho_0 \simeq 61^\circ$ is illustrated in Fig. 6. In a forthcoming paper we show that a basic role of the initial phase of positive net momentum flux $P_W^z(u)$ is to induce a large inspiral branch in the motion of the center-of-mass of the merged system.

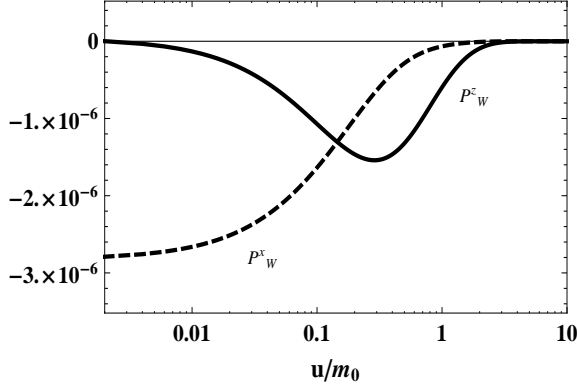


FIG. 6: Linear-log plots of the net momentum fluxes P_W^z and P_W^x carried out by the gravitational waves, for $\alpha = 0.1$, and the initial incidence angle $\rho_0 = 61^\circ$ which corresponds to a threshold configuration where the initial phase of positive net momentum flux along the z axis not present, with $P_W^z(u_0) = 0$. In this case we can see that the total impulse imparted to the merged system along the x direction is considerably larger than the impulse in the z direction.

The final momentum of the remnant can be obtained by evaluating (57) at $u = u_f$. Taking into account that the Bondi mass aspect for the final black hole remnant is given by

$$m(u, \theta, \phi) = \frac{m_0 K_f^3}{(\cosh \gamma_f + (\mathbf{n}_f \cdot \hat{\mathbf{x}}) \sinh \gamma_f)^3}, \quad (70)$$

(cf. Eqs. (12) and (47)), we obtain by a straightforward integration that

$$P_f^x = n_{1f} P_f, \quad P_f^z = n_{3f} P_f, \quad (71)$$

with modulus $P_f = m_0 K_f^3 \sinh \gamma_f$. As expected we have P_f^y .

In Fig. 7 we plot the distribution of the final momentum modulus P_f of the remnant black hole as a function of α , for several values of the incidence angle ranging

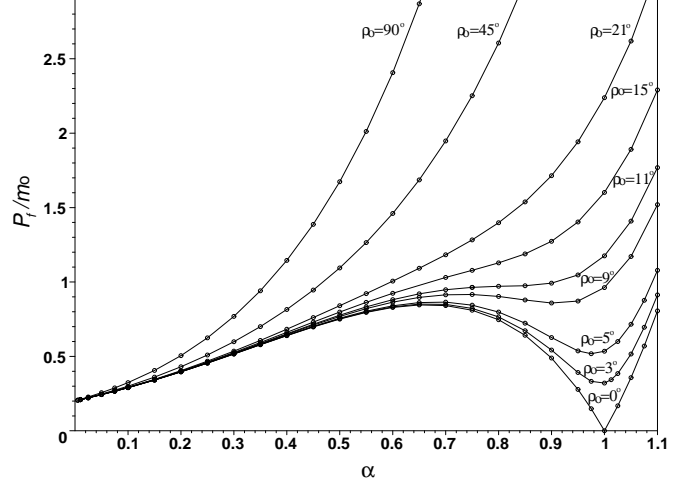


FIG. 7: Plot of the modulus P_f of the momentum of the remnant black hole as a function of the mass ratio parameter α , for several values of the incidence angle of collision ρ_0 . Points are connected for a better visualization. Values of P_f for $\alpha > 1$ are redundant due to the relation $P_f(\alpha) = \alpha^6 P_f(1/\alpha)$.

from $\rho_0 = 0$ (head-on collision) to $\rho_0 = 90^\circ$ (orthogonal collision). We see that the head-on case ($\rho_0 = 0^\circ$) is a lower bound for the remnant moment distribution. This is the only case in which the modulus of the final momentum of the remnant becomes zero, for $\alpha = 1$, as discussed in Ref. [3] where the axisymmetric dynamics was analyzed, making explicit the constrained character of a head-on collision. For sufficiently small values of the incidence angle ρ_0 we see that the behavior follows that of the head-on case, with the zero momentum at $\alpha = 1$ being replaced by a minimum at $\alpha \lesssim 1$. This minimum turns into an inflexion point for $\rho_0 \simeq 15^\circ$ and disappears for larger values of ρ_0 . Due to the dominant deceleration regime of the merged system, consequent of the linear momentum flux \mathbf{P}_W extracted by the gravitational waves, the total momentum of the system decreases towards the distribution of P_f given in Fig. 7. Finally, through an analogous analysis to the case of the efficiency Δ , we have that the final momentum P_f satisfies the relation $P_f(\alpha) = \alpha^6 P_f(1/\alpha)$ for $\alpha \in (0, \infty)$. Therefore in Fig. 7 the values of P_f for $\alpha > 1$ are redundant.

A comment is in order now. An important new feature appears in non head-on collisions connected to the final momentum for equal mass black holes ($\alpha = 1$). Contrary to the case of a head-on collision and of the merging of black hole binary inspirals, for a non head-on collision of two equal mass black holes the net gravitational wave flux emitted is nonzero. This is the reason why the final momenta in Fig. 7 for $\alpha = 1$ is nonzero, even if measured in an inertial frame with velocity $\mathbf{v}_{in} = \mathbf{P}(u_0)/M_B(u_0)$ (namely, a zero-initial-Bondi-momentum frame) relative

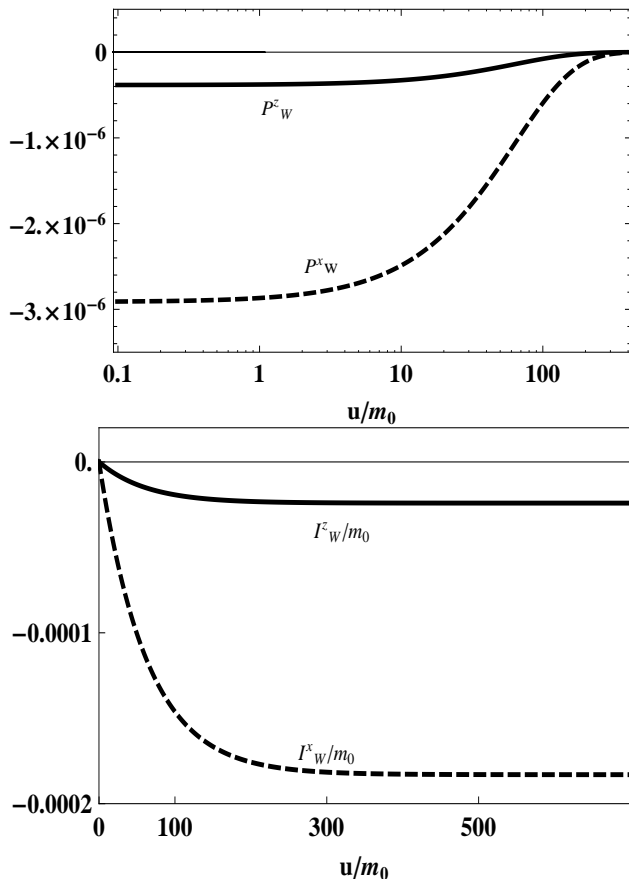


FIG. 8: Linear-log plot of the net momentum fluxes $P_W^x(u)$ and $P_W^z(u)$ (top), and the associated total impulses $I_W^x(u)$ and $I_W^z(u)$ (bottom), for the case of equal-mass initial colliding black holes ($\alpha = 1$) and incidence angle $\rho_0 = 15^\circ$. The associated kick for this case is $Vk \sim 0.9\text{km/s}$, this small value being due to the value $\gamma = 0.2$ adopted in our computation (cf. text).

to a rest inertial frame at infinity. This is illustrated in Figs. 8, where we display the net momentum flux $\mathbf{P}_W(u)$ and the associated impulse $\mathbf{I}_W(u)$ for the case of equal-mass initial colliding black holes in a non head-on collision with incidence angle $\rho_0 = 15^\circ$.

The analysis of the kick velocities generated in these processes of momentum extraction (and the associated gravitational wave recoil) will be the subject of a future paper. In a related analysis (cf. Ref. [3]) for head-on collisions we have shown that the distribution of kick velocities as a function of the symmetric mass parameter fitted nicely Fitchett's law[25] whose original derivation was based on post-Newtonian analytical estimates of gravitational wave emission in inspiral binaries. This corresponds to a Newtonian signature in the dynamics and this result may suggest that even in the head-on post-merger phase, described by RT dynamics, a component of the post-Newtonian dynamics of two interacting bodies

emitting gravitational waves might be present/preserved. In the case of non head-on collisions examined in the present paper a similar Newtonian signature appears involving the relation between the incident angle of collision ρ_0 and the scattering angle of the remnant ρ_f , as we discuss in Section X.

IX. THE ANGULAR WAVE PATTERN AND THE BREMSSTRAHLUNG REGIME OF THE GRAVITATIONAL WAVES

As discussed in Sec. II, the invariant characterization of a wave zone in RT spacetimes and the consequent presence of gravitational waves is given by the functions $D(u, \theta, \phi)$ and $B(u, \theta, \phi)$, Eqs. (8)-(9), that characterize the curvature tensor components (7) at the wave zone. These functions correspond to the two modes of polarization of the gravitational waves and contain all the information of the angular and time dependence of the gravitational wave amplitudes via the particular combination

$$D(u, \theta, \phi) + iB(u, \theta, \phi) \sim (r\Psi_4), \quad (72)$$

where Ψ_4 is the Weyl spinor associated with N_{ABCD}/r [26, 27]. According to (8)-(9), the quantity (72) is specified once we have the function $K(u, \theta, \phi) \equiv 1/P(u, \theta, \phi)$, which is in turn numerically obtained via the approximation (35).

In Fig. 9 we display the polar plots of $\sqrt{D^2 + B^2}$ at early times $u = 0.01$ (dotted), $u = 0.05$ (dash-dotted) and $u = 0.1$ (continuous), and initial data parameters $\alpha = 0.2$, $\rho_0 = 55^\circ$ and $\gamma = 0.5$, with section by the plane $\phi = 0^\circ$ (corresponding to the plane of collision $x - z$). The plots in Fig. 9 show, for each time, a pattern with two dominant lobes in the forward direction of motion of the merged system, the first quadrant of the $x - z$ plane. The direction of the Bondi momentum vector at $u = 0.01$ makes an angle $\Theta_B \simeq 8, 43^\circ$ with the z axis. The pattern is typical of a bremsstrahlung process due to the deceleration of the merged system, analogous to the electromagnetic bremsstrahlung of a charge decelerated along its direction of motion. As time increases we observe that the cone enveloping the dominant lobes opens up and the amplitudes decrease. For later times (cf. Fig. 11 for $u = 5.0$) the pattern evolves to the expected quadrupole structure with a much smaller amplitude. We mention that the increase of the initial boost parameter γ would sharpen the forward cone enveloping of the two dominant lobes in the early regime, as expected in a ultrarelativistic configuration. In our computations we fixed $m_0 = 10$. In the Figures the z direction corresponds to the vertical axis. In Fig. 10 we show the polar plot of $\sqrt{D^2 + B^2}$, with section by the plane $\phi = 90^\circ$ (corresponding to the plane $y - z$, orthogonal to the plane of collision), at $u = 0.1$. The same initial data parameters of the previous Figure were used. As expected the pattern is symmetric about the z axis in accordance with the conservation of $P_W^y(u) = 0$. We then see that, although gravitational waves are emitted outside the plane

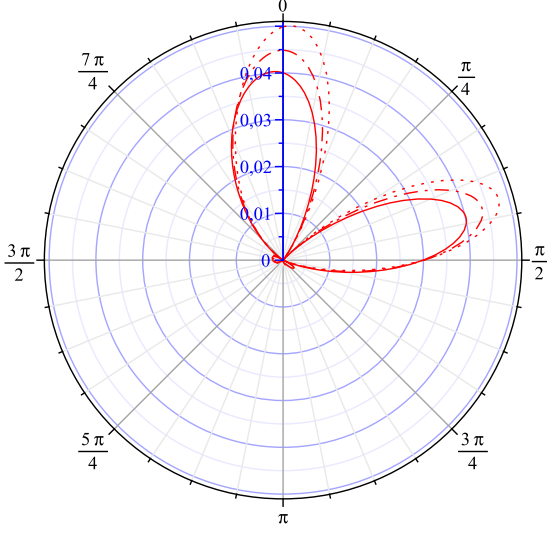


FIG. 9: Polar plot of $\sqrt{D^2 + B^2}$ (section by the plane $\phi = 0^\circ$, corresponding to the plane of the collision) for times $u = 0.01$ (dotted), $u = 0.05$ (dash-dotted) and $u = 0.1$ (continuous), and initial data parameters $\alpha = 0.2$, $\rho_0 = 55^\circ$ and $\gamma = 0.5$. The Figure shows a typical bremsstrahlung pattern, corresponding to a strong deceleration regime at early times, with two dominant lobes along the direction of motion of the merged system. The cone enveloping the two dominant lobes opens up as u increases. For larger values of γ the enveloping cone of the forward lobes becomes more sharp. The z direction corresponds to the vertical axis.

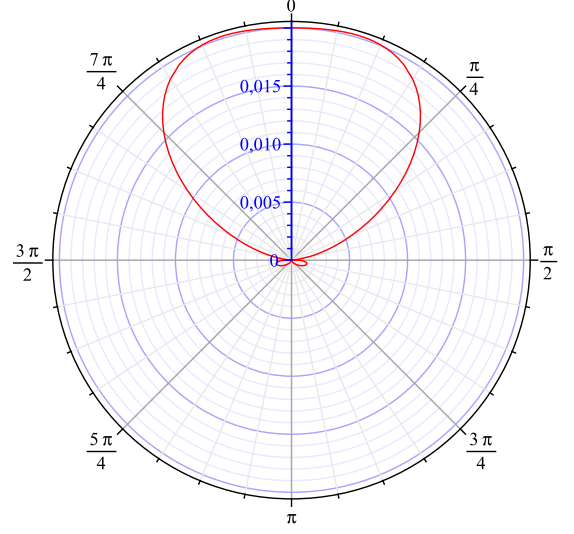


FIG. 10: Polar plot of $\sqrt{D^2 + B^2}$ (section by the plane $\phi = 90^\circ$, corresponding to the plane $y - z$, orthogonal to the plane of collision) for a time $u = 0.1$, and initial data parameters $\alpha = 0.2$, $\rho_0 = 55^\circ$ and $\gamma = 0.5$. The symmetry about the z -axis is in accordance with the conservation of $P_W^y(u) = 0$. Although gravitational waves are emitted outside the plane of collision, the zero net momentum flux of this radiation component is consistent with the planar nature of the collision.

of the collision, this radiation component has a zero net momentum flux. Therefore it does not extract momentum of the system, consistent with the planar nature of the collision.

Finally in Fig. 11 we display the polar plot of $\sqrt{D^2 + B^2}$ (section by the plane $\phi = 90^\circ$) for a time $u = 0.5$ and for the same initial data parameters of Fig. 9. We can see the opening of the dominant lobes and the forming of the final quadrupole angular pattern.

X. A NEWTONIAN SIGNATURE IN THE RELATION BETWEEN THE INCIDENCE ANGLE AND THE SCATTERING ANGLE OF THE REMNANT

One of the important parameters of the remnant black hole obtained from our numerical experiments is the angle ρ_f – defined as $\rho_f = \cos^{-1}(n_{3f})$ and denoted the scattering angle of the remnant – resulting from the evolution of initial data with initial incidence angle ρ_0 . In general, for the mass ratio parameter α sufficiently small, ρ_f is much smaller than the incidence angle ρ_0 . However for large values of α this behavior changes drastically as is the case for equal or nearly equal mass black holes.

No simple analytical relation between ρ_0 and ρ_f was

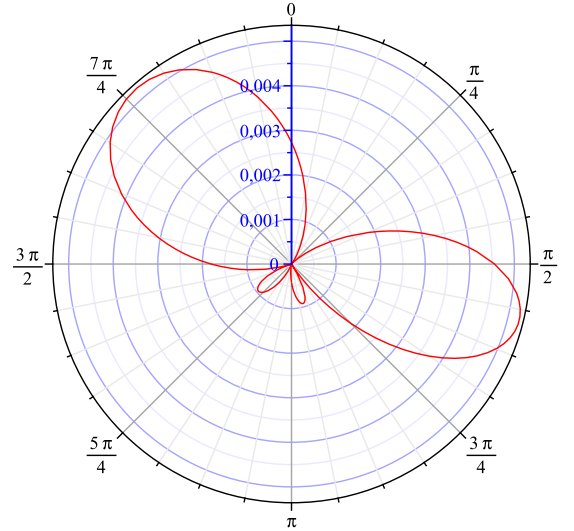


FIG. 11: Polar plot of $\sqrt{D^2 + B^2}$ (section by the plane $\phi = 0^\circ$) for a time $u = 5.0$, and same initial data parameters of the previous Figures, showing the opening of the lobes and the setting already of the final quadrupole angular pattern.

possible to be obtained from the Bondi-Sachs conservation laws. However, as we will show, the scattering angle can be related to the incidence angle by a formula that closely approximates the Newtonian relation of angles in the nonelastic scattering of classical particles, and which depends basically only on the mass ratio α . To see this let us consider the schematic diagram of collision shown in Fig. 12. There we have the two initial colliding black holes, one with total mass M boosted along the positive z axis and the other with total mass αM , with $\alpha < 1$, boosted along a direction making an angle ρ_0 with the z axis, in accordance with the asymptotic result (23). In this picture we consider the approximation in which we neglect the contribution of the initial gravitational wave content to the individual masses, although this contribution is small and in any case can be isolated. The direction of the momentum of the remnant black hole makes an angle ρ_f (the scattering angle) with the z axis. Furthermore, although in RT dynamics the presence a global apparent horizon is already present at the initial time and the system may be viewed as a deformed black hole, we are also assuming that inside the event horizon we still have two individual colliding black holes.

Under the above assumptions and taking into account that the dynamics of the collision involves mass and momentum loss due to the emission of gravitational waves, the balance of linear momentum can then be approximately expressed as

$$\begin{aligned} M \sinh \gamma (1 - \alpha \cos \rho_0) &= (\delta_z) m_0 K_f^3 \cos \rho_f \sinh \gamma_f, \\ M \sinh \gamma (\alpha \sin \rho_0) &= (\delta_x) m_0 K_f^3 \sin \rho_f \sinh \gamma_f, \end{aligned}$$

where δ_z and δ_x are correction factors introduced to account for the mass and momentum loss in the process. The ratio of the above equations yield the relation

$$\frac{\alpha \sin \rho_0}{1 - \alpha \cos \rho_0} = \delta \tan \rho_f \quad (73)$$

where $\delta \equiv (\delta_x/\delta_z)$. Eq. (73) provides a relation between the initial incidence angle and the scattering angle of the remnant and must be validated by the results of the numerical experiments with the initial data (53). We note that (73) reproduces the equation for the inelastic collision of classical particles when $\delta = 1$.

Numerical results for ρ_f obtained from initial data sampled in the whole range $0 < \alpha \leq 1$ shows that the parameter $\delta \sim 1$, its discrepancy from the Newtonian classical value 1 occurring for relatively small values of α , where the asymmetry in the data is large. For illustration

$$\begin{aligned} (\alpha &= 0.1, \quad \rho_0 = 45^\circ, \quad \rho_f \simeq 4.16^\circ, \quad \delta \simeq 1.046047) \\ (\alpha &= 0.6, \quad \rho_0 = 45^\circ, \quad \rho_f \simeq 35.78^\circ, \quad \delta \simeq 1.022573) \\ (\alpha &= 1.0, \quad \rho_0 = 45^\circ, \quad \rho_f \simeq 67.50^\circ, \quad \delta \simeq 1.000000) \end{aligned}$$

As α increases $\delta \rightarrow 1$, the limiting value $\delta = 1$ being attained for $\alpha = 1.0$ and any ρ_0 , yielding consequently (for $\alpha = 1$)

$$\rho_f = \frac{(180^\circ - \rho_0)}{2}, \quad (74)$$

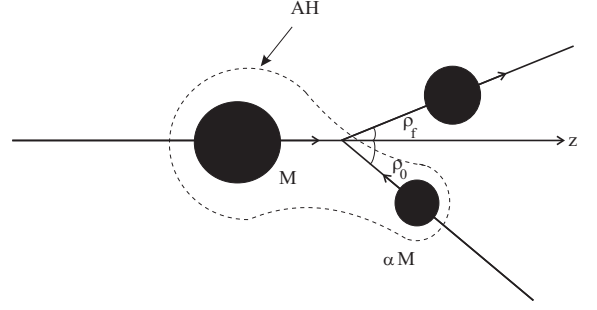


FIG. 12: Schematic diagram of collision of two boosted black holes with initial infalling velocity parameter γ and mass ratio α . The dashed line depicts the common apparent horizon enclosing the two colliding black holes.

with error of the order of, or smaller than 10^{-8} , for all experiments made with $\alpha = 1$. In fact, $\rho_f = (180^\circ - \rho_0)/2$ is an exact solution for the Newtonian inelastic collision equation (namely, Eq. (73) with $\delta = 1$). It is also worth remarking that for equal-mass initial colliding black holes ($\alpha = 1$) the direction determined by the angle $(180^\circ - \rho_0)/2$ with the z axis corresponds to an axis of symmetry of the initial data (cf. Fig. 12).

Finally it can be of physical interest to compare the scattering angle with the initial momentum direction. In our above approximation of the momentum conservation we obtain, by a straightforward calculation, that the angle Θ_0 of the initial momentum direction with the z -axis is given by $\tan \Theta_0 \equiv P^x(u_0)/P^z(u_0) = \alpha \sin \rho_0 / (1 - \alpha \cos \rho_0)$, resulting from the relation (73) that $\tan \Theta_0 = \delta \tan \rho_f$. This shows that the direction of motion of the remnant presents a very small deviation from the direction of the initial momentum θ_0 , as seen from an asymptotic inertial observer. This small difference is connected to the small values of the impulse imparted to the merged system by the gravitational waves emitted. However these small values of the impulse will be sufficient to generate kicks of a few hundred km/s in the system.

The above results, relating ρ_0 and ρ_f , suggest a Newtonian signature present in the dynamics of the merged system with initial data (53), indicating that the individuality of the two initial colliding black holes is in some sense preserved in the initial deformed merged system.

XI. CONCLUSIONS AND FINAL REMARKS

In this paper we have examined the numerical evolution of characteristic initial data corresponding to non head-on collisions of two Schwarzschild black holes, in the realm of Robinson-Trautman non-axisymmetric spacetimes. The Robinson-Trautman spacetimes present already a global apparent horizon so that the dynamics covers a regime where the merger has already set out. We have constructed initial data for the characteristic surface

formalism of Robinson-Trautman dynamics, that represent instantaneously this system. The RT equation is integrated numerically using a Galerkin spectral method with a projection basis space in two variables (the spherical harmonics). The planar nature of a generic non-head-on collision of two black holes, which is restricted to the plane determined by the two directions of motion of the two initial colliding black holes, is used to test the accuracy of our numerical method and allows us to save a lot of computational effort. The three basic independent parameters characterizing the initial data are the mass ratio parameter α , the boost parameter γ (that defines the initial infalling velocity $v = \tanh \gamma$ of the initial individual black holes) and the incidence collision angle ρ_0 of the two initial colliding black holes.

During the merger gravitational waves are emitted, extracting mass and linear momentum of the system, until the remnant configuration is reached, when the gravitational wave emission ceases. Typically the remnant is a boosted Schwarzschild black hole defined by its final rest mass $m_0 K_f^3$, its final velocity parameter $v_f = \tanh \gamma_f$ and its direction of motion determined by the remnant scattering angle ρ_f .

We evaluate the efficiency Δ of the mass-energy extraction as a function of the mass ratio parameter α for several values of the incidence angle ρ_0 . We obtain that head-on collisions ($\rho_0 = 0^\circ$) constitute an upper bound, while orthogonal collisions ($\rho_0 = 90^\circ$) constitute a lower bound for the efficiency. This fact is connected to the behavior of the total gravitational wave energy emitted in the process since the head-on collision and the orthogonal collision are configurations that correspond, respectively, to an upper bound and a lower bound for the total energy emitted. The efficiency presents a maximum at $\alpha = 1$ for all values of the incidence angle, $0^\circ \leq \rho_0 \leq 90^\circ$, and for any boost parameter γ , which is shown to be a consequence of some symmetries of the initial data (53).

The analysis of the momentum and energy flux of the gravitational waves is made using the Bondi-Sachs energy-momentum conservation laws. The momentum extraction by gravitational waves with the consequent recoil of the system is evaluated, with respect to an asymptotic Lorentz frame. For incidence angles ρ_0 up to a certain threshold value, we have that the net momentum flux carried out by the gravitational waves along the z axis is positive for a short initial phase leading to an increase of the momentum of the merged system along this direction, $dP^z(u)/du > 0$. This phase is followed by a regime with negative net momentum flux that lasts until the final remnant configuration is attained. For ρ_0 larger than the threshold value the initial phase of positive momentum flux is no longer present. For $\alpha = 0.1$ the threshold angle is $\rho_0 \simeq 61^\circ$. Relative to the asymptotic Lorentz frame the Bondi-Sachs momentum components $P^z(u)$ and $P^x(u)$ are positive for all u .

On the other hand, the net momentum flux along the x direction is negative, for all times u and incidence angle ρ_0 , so that the momentum of the system along the x

direction always decreases. We have that $P_W^y(u) = 0$ for all u , a consequence of the planar nature of the collision, so that the momentum of the merged system along this direction is conserved (in fact we have $P^y(u) = 0$ for all u).

In general we have a net strong deceleration of the system due to the emission of gravitational waves, as a consequence of the the total impulse of the gravitational waves corresponding to the integrated fluxes. Typically, for large $u \sim u_f$, the curves of the impulse tend to a negative constant value (a plateau) associated with the final configuration when the gravitational wave emission ceases. As a consequence, in accordance with the impulse conservation (61) for $\Omega = u_f$, there is a net momentum decrease due to the total impulse of the gravitational waves imparted to the binary merged system. The importance of the initial acceleration of the z -component of the momentum (when present) is to produce a large inspiral branch in the motion of the center-of-mass of the system, as seen from the initial zero momentum frame, as we will discuss in a future publication.

The momentum distribution of the remnant P_f/m_0 was evaluated for several ρ_0 in the range $0^\circ \leq \rho_0 \leq 90^\circ$. It has the head-on collision case as a lower bound and the orthogonal collision as an upper bound, for all α . A new feature is observed in the final momentum distribution for equal-mass black holes, in the case of non head-on collisions. Contrary to the case of a head-on collision and of the merging of black hole binary inspirals, for a non head-on collision of two equal mass black holes the net gravitational wave flux emitted is nonzero. Therefore the final momentum P_f for $\alpha = 1$ is nonzero, for any $\rho_0 \neq 0$, even if measured in an inertial frame with velocity $\mathbf{v}_{in} = \mathbf{P}(u_0)/M_B(u_0)$ (the zero initial-Bondi-momentum frame) relative to a rest inertial frame at infinity. The issue of kick velocities of the merged system due to the recoil of gravitational wave emission is currently being examined. Our analysis is based on the impulse conservation equations (61) and we obtain that, in a zero initial Bondi momentum frame, the velocity of the center-of-mass at $u = u_f$ coincides approximately with the kick velocity. The distribution of the kick velocity as a function of the symmetric mass parameter satisfies a modified Fitchett-Blanchet law[25]. This modification is necessary to give account of the fact that, in a non head-on collision of two equal-mass black holes, the net gravitational wave flux emitted is nonzero, contrary to the case of a head-on collision and of merging of black hole inspirals.

The angular pattern of the gravitational waves emitted (with its two modes of polarization included) is examined for the case of a low mass ratio parameter and a large incidence angle. In the plane of the collision the pattern is typically bremsstrahlung, with two dominant lobes in the forward direction of motion of the merged system, characteristic of a strong initial decelerated regime of the merged system. We show that gravitational waves are also emitted outside the plane of the collision but that this radiation has a zero net momentum flux. Therefore it

does not extract momentum of the system, in accordance with the planar nature of the collision.

Finally we also obtained a relation between the initial incidence angle ρ_0 and the scattering angle ρ_f that closely reproduces a result for the inelastic collision of classical particles in Newtonian dynamics, and is validated by the numerical results. For $\alpha = 1$ we recover numerically the exact Newtonian result, with the $\rho_f = (180^\circ - \rho_0)/2$ being a consequence of the symmetry of the initial data as if the two initial black holes and the remnant were individ-

ual classical particles. This result suggests a Newtonian signature present in the dynamics of the merged system with initial data (53).

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